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Gepner-Like Description of a String Theory on a Non-compact Singular Calabi-Yau Manifold

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Abstract: We investigate a Gepner-like superstring model described by a combination of multiple minimal models and an $\mathcal{N} = 2$ Liouville theory. This model is thought to be equivalent to the superstring theory on a singular non-compact Calabi-Yau manifold. We construct the modular invariant partition function of this model, and confirm the validity of an appropriate GSO projection. We also calculate the elliptic genus and Witten index of the model. We find that the elliptic genus factorises into a rather trivial factor and a non-trivial one, and the non-trivial one has the information on the positively curved base manifold of the cone.

1 Introduction

The correspondence between a string theory on a Kähler manifold and an $\mathcal{N}=2$ Landau-Ginzburg theory is interesting and is very largely investigated [1, 2, 3, 4]. But, the most results are limited to the cases of compact Calabi-Yau manifolds. Recently, it is conjectured that in the case of a non-compact Calabi-Yau manifold, the associated CFT consists of an $\mathcal{N}=2$ Liouville theory and a Landau-Ginzburg theory [5]. They claimed that when the Calabi-Yau n-fold X is written as a hypersurface $F(z_1,\ldots,z_{n+1})=0$ in $\mathbf{C}^{n+1}$ by a quasi-homogeneous polynomial F, then the string theory on X is equivalent to the CFT on

$$R_\phi \times S^1 \times LG(W=F).$$

Here R_ϕ is a real line parametrized by ϕ with linear dilaton background and $LG(W=F)$ is the IR theory of the Landau-Ginzburg model with the superpotential F. In this case, the background charge Q of R_ϕ is determined by a condition of the total central charge. From the condition that Q is a non-zero real number, we find that the base manifold $X/\mathbf{C}^\times$ should be curved positively.

The boson with linear dilaton background is strongly coupled in the region $\phi\to-\infty$, so we should introduce the Liouville potential or consider an $SL(2)/U(1)$ Kazama-Suzuki model to avoid the strong coupling singularity [6, 7, 8]. But we do not care about this point in this paper.

In [9, 5, 7, 8], they also claim that the string theory on this singular non-compact Calabi-Yau manifold is holographic dual to the "little string theory".

In the case of a compact Calabi-Yau manifold, the string theory is "solved" in the description of Gepner model in a special point of the moduli space. We want to describe also the string theory the non-compact Calabi-Yau manifold by the Gepner-like solvable model. If we can do it successfully, it will be possible to analyze more deeply a non-compact Calabi-Yau manifold and the little string theory.

In [10], they treat the string theories with ADE simple singularities. They construct the modular invariant partition functions, and show the consistency of these string theories.

In this paper, we consider more general cases, in which the Landau-Ginzburg part is described by a direct product of a number of minimal models. A typical example of ours is the Calabi-Yau n-fold described in the form

$$z_1^{N_1}+z_2^{N_2}+\cdots+z_{n+1}^{N_{n+1}}=0,\ \text{in } \mathbf{C}^{n+1}.$$

We construct the modular invariant partition functions and show the string theory actually exists consistently in these cases. We also calculate the elliptic genus, and find that it factorises into two factors — a rather trivial one and a rather non-trivial one. We analyze the non-trivial one in detail, and find that it has the information on the cohomology of the base manifold $X/\mathbf{C}^\times$ except the elements generated by cup products of a Kähler form.

The organization of this paper is as follows. In the next section, we explain the setup and review the correspondence between a non-compact Calabi-Yau manifold and an $\mathcal{N}=2$ Liouville theory $\times$ Landau-Ginzburg theory. In section 3, we construct the modular invariant partition function. In section 4, we calculate the elliptic genus and compare it with the geometric property of the associated Calabi-Yau manifold X. In the last section, we summarize the results and discuss the problems and prospects. In Appendix A. we collect some useful equations of theta functions and characters that we use in this paper.

2 The String theory on a non-compact singular Calabi-Yau manifold

We consider the string compactification to a non-compact, singular Calabi-Yau n-fold X. The total target space is expressed by a direct product of a d dimensional flat spacetime and the manifold X

$$\mathbf{R}^{d-1,1} \times X.$$

Here, n is related to d by the constraint on the total dimension $2n+d=10$.

For simplicity, we concentrate the case that the non-compact singular Calabi-Yau manifold is realized as the hypersurface in $\mathbf{C}^{n+1}$ determined by the algebraic equation with a quasi homogeneous polynomial F

$$F(z_1,\dots,z_{n+1})=0.$$

By the term "quasi-homogeneous", we mean that the polynomial F satisfies

$$F(\lambda^{r_1}z_1,\dots,\lambda^{r_{n+1}}z_{n+1})=\lambda F(z_1,\dots,z_{n+1}), \tag{2.1}$$

for some exponents $\{r_j\}$ and for an arbitrary $\lambda\in\mathbf{C}^{\times}$.

It is conjectured in [5] that the string theory mentioned above is equivalent to the theory including flat spacetime $\mathbf{R}^{d-1,1}$, a line with the linear dilation background R_ϕ, S^1, and the Landau-Ginzburg theory with a super potential $W=F$;

$$\mathbf{R}^{d-1,1}\times\mathbf{R}_\phi\times S^1\times LG(W=F).$$

The part $(\mathbf{R}_\phi\times S^1)$ has a world sheet $\mathcal{N}=2$ superconformal symmetry. Let ϕ be the parameter of $\mathbf{R}_\phi$, Y be the parameter of S^1. And let ψ^+,ψ^- be the fermionic part of $(\mathbf{R}_\phi\times S^1)$. The $\mathcal{N}=2$ superconformal currents are written in terms of the above fields

$$\begin{aligned}
T &= -\frac{1}{2}(\partial Y)^2-\frac{1}{2}(\partial\phi)^2-\frac{Q}{2}\partial^2\phi-\frac{1}{2}(\psi^+\partial\psi^- - \partial\psi^+\psi^-),\\
G^\pm &= -\frac{1}{\sqrt{2}}\psi^\pm(i\partial Y\pm\partial\phi)\mp\frac{Q}{\sqrt{2}}\partial\psi^\pm,\\
J &= \psi^+\psi^- - Qi\partial Y.
\end{aligned} \tag{2.2}$$

The associated central charge of this algebra is $\hat{c}(=c/3) = 1 + Q^2$.

In this paper, we consider the case in which the Landau-Ginzburg theory with superpotential $W = F$ can be described by a direct product of $\mathcal{N} = 2$ minimal models. Let $M_{G,N}$ be the minimal model corresponding to simply laced Lie algebra $G = A, D, E$ with dual Coxeter number N. We consider the theory in the following;

$$\mathbf{R}^{d-1,1} \times \mathbf{R}_\phi \times S^1 \times M_{G_1,N_1} \times \cdots \times M_{G_R,N_R},$$

where R is the number of the minimal models. The cases with $R = 1$ are treated in [10] and $R = 0$ in [11, 12]

A typical example is the case that all G_j are A type. In this example, the quasi-homogeneous polynomial is written as

$$F(z) = z_1^{N_1} + \cdots + z_R^{N_R} + z_{R+1}^2 + \cdots + z_{n+1}^2.$$

The background charge of $\mathbf{R}_\phi$ is determined by the criticality condition. To cancel the conformal anomaly, the total central charge is to be 0. The central charge of the ghost sector is -15, so the total central charge of the matter sector is to be 15. The central charge of the flat spacetime is 3/2 for each pair of a boson and a fermion, and that of the $\mathbf{R}_\phi \times S^1$ is $3 + 3Q^2$ as mentioned above, and that of a minimal model $M_{G,N}$ is $\frac{3(N-2)}{N}$. Therefore, the criticality condition leads to the equation

$$\frac{3d}{2} + 3 + 3Q^2 + \sum_{j=1}^{R} \frac{3(N_j - 2)}{N_j} = 15.$$

From this criticality condition, we obtain the value of Q^2 as

$$Q^2 = 4 - \frac{d}{2} - \sum_j \frac{3(N_j - 2)}{N_j}. \tag{2.3}$$

By the condition $Q^2 > 0$ for a real number Q, the right-hand side should be positive;

$$4 - \frac{d}{2} - \sum_j \frac{3(N_j - 2)}{N_j} > 0. \tag{2.4}$$

It is equivalent to a condition that the singularity is in finite distance in the Calabi-Yau moduli space[5, 13]. In the point of view of the $\mathbf{C}^\times$ action defined in (2.1) , the finite distance condition is equivalent to that $X/\mathbf{C}^\times$ is positively curved.

3 Modular invariant partition function

Now, let us construct the modular invariant partition function. We take the light-cone gauge, then the associated CFT to consider is

$$\mathbf{R}^{d-2} \times \mathbf{R}_\phi \times S^1 \times M_{G_1,N_1} \times \cdots \times M_{G_R,N_R}.$$

The toroidal partition function can be separated into 2 parts: the one Z_{GSO} concerning to the GSO projection and the other Z_0 not concerning to it. We construct the total partition function Z as

$$Z = \int \frac{d^2\tau}{\tau_2^2} Z_0(\tau, \bar{\tau}) Z_{GSO}(\tau, \bar{\tau}),$$

where $\tau = \tau_1 + i\tau_2$ is the moduli parameter of the torus, and $d^2\tau/\tau_2^2$ is the modular invariant measure.

First, we study the rather easy part Z_0, then we investigate the rather complicated part Z_{GSO}.

3.1 GSO independent part of the partition function

In this subsection, we discuss the Z_0 : the GSO independent part. It is completely the same as that in [10].

The Z_0 includes the contribution from the flat spacetime bosonic coordinates X^I, $(I = 2, \ldots, d-1)$ and the linear dilation ϕ.

The partition function of for each flat spacetime boson is represented by the Dedekind eta function $\eta(\tau)$ as

$$\frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}.$$

The partition function of ϕ is defined as $Z_L = \mathrm{Tr} q^{L_0 - c_L/24} \bar{q}^{\bar{L}_0 - c_L/24}$, $(q = \exp(2\pi i\tau), c_L = 1 + 3Q^2)$ in the canonical formalism. Here the trace is taken over delta functional normalizable primary fields

$$\exp(ip\phi), \quad p = -\frac{iQ}{2} + \ell, \quad \ell \in \mathbf{R}, \tag{3.1}$$

and its excitation by oscillators. Then we obtain Z_L as

$$\begin{aligned} Z_L &= \frac{1}{|\prod_{n=1}(1-q^n)|^2} \int dp \exp\left[-4\pi\tau_2\left(\frac{1}{2}p^2 + \frac{i}{2}pQ - \frac{1+3Q^2}{24}\right)\right] \\ &= \frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}, \end{aligned}$$

where the region of the integral of p is as (3.1). As a result, the partition function of ϕ is the same as that of a ordinary boson. So we obtain Z_0 as the partition function of effectively $(d-1)$ free bosons;

$$Z_0 = \left(\frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}\right)^{d-1}.$$

The primary fields of (3.1) correspond to "Principal continuous series" in terms of the representation of $SL(2)$. To include the other sectors is an interesting problem and is postponed as a future work.

3.2 GSO dependent part of the partition function : $d = 2, 6$ cases

Now, let us proceed to the GSO dependent part Z_{GSO}. In this subsection, we treat $d = 2, 6$ cases.

This part includes $(d-2)$ flat spacetime fermions ψ^I, $(I = 2, \ldots, d-1)$, two free fermions ψ^ϕ, ψ^Y associated to $\mathbf{R}_\phi \times S^1$, minimal models $M_{G_1,N_1}, \ldots, M_{G_R,N_R}$, and an S^1 boson Y. We combine the d free fermions ψ^I, $(I = 2, \ldots, d-1), \psi^\phi$ and ψ^Y and construct the affine Lie algebra $\widehat{SO(d)}_1$. The Verma module of $\widehat{SO(d)}_1$ is characterized by an integer $s_0 = 0, 1, 2, 3$, which labels the representations of $SO(d)$,that is scalar, spinor, vector, and cospinor, respectively.

Let us turn to Verma modules of $\mathcal{N} = 2$ minimal models. The Verma module of an $\mathcal{N} = 2$ minimal model is specified with three indices (ℓ, m, s), which satisfy the following conditions[2]

$$
\begin{aligned}
&\ell = 0, \ldots, N-2, \\
&m = 0, \ldots, 2N-1, \\
&s = 0, 1, 2, 3, \\
&\ell + m + s \equiv 0 \mod 2.
\end{aligned}
\tag{3.2}
$$

We denote $\chi_m^{\ell,s}(\tau, z)$ as the character of the Verma module labeled by the set (ℓ, m, s). Some properties of this character is collected in Appendix A.

The Verma module of the whole GSO dependent parts is specified by the index s_0 of $\widehat{SO(d)}_1$ representation, the indices (ℓ_j, m_j, s_j), $(j = 1, \ldots, R)$ of the minimal models, and the S^1 momentum p. We combine these indices except p into two vectors λ, μ.

$$
\begin{aligned}
\lambda &:= (\ell_1, \ldots, \ell_R), \\
\mu &:= (s_0; s_1, \ldots, s_R; m_1, \ldots, m_R).
\end{aligned}
$$

We shall introduce the inner product between μ and μ' vectors as in [2]

$$
\mu \bullet \mu' := -\frac{d}{2}\frac{s_0 s_0'}{4} - \sum_{j=1}^{R} \frac{s_j s_j'}{4} + \sum_{j=1}^{R} \frac{m_j m_j'}{2N_j}.
$$

Also it is convenient to introduce special vectors β_0, β_j $(j = 1, \ldots, R)$

$$
\begin{aligned}
\beta_0 &:= (1; 1, \ldots, 1; 1, \ldots, 1), \\
\beta_j &:= (2; 0, \ldots, 0, \underset{\overset{\wedge}{s_j}}{2}, 0, \ldots 0; 0, \ldots, 0).
\end{aligned}
$$

Here the β_0 is the vector with all components 1, and β_j is the vector with s_0 and s_j components 2 and the others zero.

With these notations, the criticality condition (2.3) can be written in a rather simple form as

$$Q^2 = 4(1 + \beta_0 \bullet \beta_0). \tag{3.3}$$

When we define an integer $K := \mathrm{lcm}(2, N_j)$, KQ^2 is shown to be an even integer because of the equations (2.3) and (3.3). Therefore, it is convenient to define an integer J by the equation

$$J := 2K(1 + \beta_0 \bullet \beta_0) \quad (= KQ^2/2). \tag{3.4}$$

In terms of J, the finite distance condition (2.4) can be expressed as $J > 0$.

Now, let us consider the character of the Verma module (λ, μ, p),

$$\chi_\mu^\lambda(\tau) \frac{q^{\frac{1}{2}p^2}}{\eta(\tau)},$$

where $\chi_\mu^\lambda(\tau)$ is the product of characters of the minimal models and the $\widehat{SO(d)}_1$ character $\chi_{s_0}(\tau)$ of the s_0 representation

$$\chi_\mu^\lambda(\tau) := \chi_{s_0}(\tau)\chi_{m_1}^{\ell_1}(\tau) \ldots \chi_{m_R}^{\ell_R}(\tau).$$

In this character, $\chi_\mu^\lambda(\tau)$ has good modular properties, but $q^{\frac{1}{2}p^2}/\eta(\tau)$ has bad ones. So, we will sum up the characters with respect to certain values of p and make the modular properties good [11][10].

Let us consider the GSO projection. By the GSO projection, We pick up the states with odd integral $U(1)$ charges of the $\mathcal{N} = 2$ superconformal symmetry. The $U(1)$ charge of the states in the above Verma module is expressed as

$$2\beta_0 \bullet \mu + pQ = -\frac{d}{2}\frac{s_0}{2} - \sum_j \frac{s_j}{2} + \sum_j \frac{m_j}{N_j} + pQ.$$

From the condition that this $U(1)$ charge must be an odd integer (2u+1) with $u \in \mathbf{Z}$, the S^1 momentum p is written as

$$p(u) = \frac{1}{Q}\left(2u + 1 - 2\beta_0 \bullet \mu\right).$$

If we sum up the characters for all $u \in \mathbf{Z}$, we obtain the theta function with a fractional level[10], which does not have good modular properties. So we perform the following trick.

Let us write $u = Jv + w$ with integers v, w and sum up the characters for $v \in \mathbf{Z}$. Then the sum leads to the following theta function

$$\sum_{v \in \mathbf{Z}} q^{\frac{1}{2}p(u=Jv+w)^2} = \Theta_{-2K\beta_0 \bullet \mu + K(2w+1), KJ}(\tau). \tag{3.5}$$

Note that $-2K\beta_0 \bullet \mu + K(2w+1)$ is an integer, and the above theta function has good modular properties.

Now, including oscillator modes and other sectors, we can define the building blocks $f^{\lambda}_{\underset{\sim}{\mu}}(\tau)$ by

$$f^{\lambda}_{\underset{\sim}{\mu}}(\tau) := \chi^{\lambda}_{\mu}(\tau)\Theta_{M,KJ}(\tau)/\eta(\tau),$$
$$\underset{\sim}{\mu} := (\mu, M), \;\; M \in \mathbf{Z}_{2KJ}.$$

We should use only the building blocks $f^{\lambda}_{\underset{\sim}{\mu}}$ with the conditions

$$M = -2K\beta_0 \bullet \mu + K(2w+1), \;\; \text{for } {}^{\exists}w \in \mathbf{Z}, \tag{3.6}$$
$$s_0 \equiv s_1 \equiv \cdots \equiv s_R \mod 2. \tag{3.7}$$

The condition (3.6) comes from the form of (3.5), and the condition (3.7) implies that the boundary condition of the fermionic currents are the same in all the sub-theories, i.e. they must be all in the NS sector, or all in the R sector.

The modular invariant partition function can be systematically obtained by "the beta method"[2].

The inner product between of two vectors $\underset{\sim}{\mu}, \underset{\sim}{\mu}'$ is defined as

$$\underset{\sim}{\mu} \bullet \underset{\sim}{\mu}' := \mu \bullet \mu' - \frac{MM'}{2KJ}.$$

We also extend the vectors β_0, β_j to $\underset{\sim}{\beta_0}, \underset{\sim}{\beta_j}$ as

$$\underset{\sim}{\beta_0} := (\beta_0, -J),$$
$$\underset{\sim}{\beta_j} := (\beta_j, 0),$$

and evaluate the inner products these $\underset{\sim}{\beta_0}, \underset{\sim}{\beta_j}$ vectors

$$\underset{\sim}{\beta_0} \bullet \underset{\sim}{\beta_0} = \beta_0 \bullet \beta_0 - \frac{J^2}{2KJ} = -1,$$
$$\underset{\sim}{\beta_j} \bullet \underset{\sim}{\beta_j} = \beta_j \bullet \beta_j = -\frac{d}{2} - 1,$$
$$\underset{\sim}{\beta_j} \bullet \underset{\sim}{\beta_0} = \frac{1}{2}\left(-\frac{d}{2} - 1\right). \tag{3.8}$$

Note that $\underset{\sim}{\beta_0} \bullet \underset{\sim}{\beta_0}$ is an odd integer, $\underset{\sim}{\beta_j} \bullet \underset{\sim}{\beta_j}$ are even integers (Recall that we consider the cases $d = 2, 6$), and $\underset{\sim}{\beta_j} \bullet \underset{\sim}{\beta_0}$ are integers. Using these special vectors, the conditions (3.6) and (3.7) are written in a simple form

$$2\underset{\sim}{\beta}_0 \bullet \underset{\sim}{\mu} \in 2\mathbf{Z}+1,$$
$$\underset{\sim}{\beta}_j \bullet \underset{\sim}{\mu} \in 2\mathbf{Z}. \tag{3.9}$$

We call this condition "the beta condition".

Using these notations, and the modular transformation laws of theta functions and $\mathcal{N}=2$ characters written in the Appendix A, we can calculate the modular transformation laws of $f^{\lambda}_{\underset{\sim}{\mu}}$ as

$$f^{\lambda}_{\underset{\sim}{\mu}}(\tau+1) = \mathbf{e}\left[\sum_j \frac{\ell_j(\ell_j+1)}{4N_j} - \frac{1}{2}\underset{\sim}{\mu}\bullet\underset{\sim}{\mu} - \frac{1}{24}\left(\sum_j \frac{N_j-2}{N_j} + \frac{d}{2}+1\right)\right] f^{\lambda}_{\underset{\sim}{\mu}}(\tau),$$
$$f^{\lambda}_{\underset{\sim}{\mu}}(-1/\tau) = \sum_{\lambda',\underset{\sim}{\mu}'}^{\text{even}} A_{\lambda\lambda'}\left(\prod_j \frac{1}{2\sqrt{N_j}}\right)\frac{1}{\sqrt{2KJ}}\,\mathbf{e}\left[\underset{\sim}{\mu}\bullet\underset{\sim}{\mu}'\right] f^{\lambda'}_{\underset{\sim}{\mu}'}(\tau),$$

where the sums of the λ', μ' are taken only for the range (3.2) and for $M = 0,\ldots,2KJ-1$. Especially we must impose the condition $\ell_j + m_j + s_j \equiv 0 \mod 2$ for each the minimal model. $A_{\lambda\lambda'}$ is the products of the $\widehat{SU(2)}_{N_j-2}$ S matrices $A^{(N_j)}_{\ell_j\ell'_j}$;

$$A_{\lambda\lambda'} = \prod_j A^{(N_j)}_{\ell_j\ell'_j} = \prod_j \sqrt{\frac{2}{N_j}}\sin\pi\frac{(\ell_j+1)(\ell'_j+1)}{N_j},$$

and we use here and the rest of this paper the notation $\mathbf{e}[x] = \exp(2\pi i x)$.

Let us note that if a vector $\underset{\sim}{\mu}$ satisfies the beta condition (3.9), the vector $\underset{\sim}{\mu} + b_0\underset{\sim}{\beta}_0 + \sum_j b_j\underset{\sim}{\beta}_j$ for $b_0, b_j \in \mathbf{Z}$, $(j = 1,\ldots,R)$ also satisfies the beta condition by virtue of (3.8). Using this fact, we define the function $F^{\lambda}_{\underset{\sim}{\mu}}$ for $(\lambda, \underset{\sim}{\mu})$ which satisfies the beta conditions (3.9) as a sum of $f^{\lambda}_{\underset{\sim}{\mu}+b_0\underset{\sim}{\beta}_0+\sum_j b_j\underset{\sim}{\beta}_j}$'s as

$$F^{\lambda}_{\mu}(\tau) = \sum_{b_0,b_j}(-1)^{s_0+b_0} f^{\lambda}_{\underset{\sim}{\mu}+b_0\underset{\sim}{\beta}_0+\sum_j b_j\underset{\sim}{\beta}_j}(\tau),$$

where the sum is taken for $b_0 \in \mathbf{Z}_{2K}$ and $b_j \in \mathbf{Z}_2$. The sign $(-1)^{s_0+b_0}$ is (-1) for the Ramond sector.

These functions are very good modular properties, especially by S transformation, the functions are mixed among those which satisfy the beta condition;

$$F^{\lambda}_{\underset{\sim}{\mu}}(\tau+1) = \mathbf{e}\left[\sum_j \frac{\ell_j(\ell_j+1)}{4N_j} - \frac{1}{2}\underset{\sim}{\mu}\bullet\underset{\sim}{\mu} - \frac{1}{24}\left(\sum_j \frac{N_j-2}{N_j} + \frac{d}{2}+1\right)\right] F^{\lambda}_{\underset{\sim}{\mu}}(\tau),$$
$$F^{\lambda}_{\underset{\sim}{\mu}}(-1/\tau) = \sum_{\lambda',\underset{\sim}{\mu}'}^{\text{even,beta}} A_{\lambda\lambda'}\left(\prod_j \frac{1}{2\sqrt{N_j}}\right)\frac{1}{\sqrt{2KJ}} e[\underset{\sim}{\mu}\bullet\underset{\sim}{\mu}'](-1)^{s_0+s'_0} F^{\lambda'}_{\underset{\sim}{\mu}'}(\tau),$$

where the sums of $\lambda', \underset{\sim}{\mu}'$ is taken for restricted subclass that satisfy the conditions (3.2) and the beta condition (3.9).

With this function $F^{\lambda}_{\underset{\sim}{\mu}}$, we obtain the modular invariant Z_{GSO} as

$$Z_{GSO}(\tau,\bar{\tau}) = \frac{1}{4^R \times 2K} \sum_{\lambda,\bar{\lambda},\underset{\sim}{\mu}}^{\text{even,beta}} L_{\lambda\bar{\lambda}} F^{\lambda}_{\underset{\sim}{\mu}}(\tau) \bar{F}^{\bar{\lambda}}_{\underset{\sim}{\mu}}(\bar{\tau}),$$

where $L_{\lambda\bar{\lambda}} = \prod_j L^{(G_j,N_j-2)}_{\ell_j\bar{\ell}_j}$ is the products of $G_j = A, D, E$ type modular invariants of $\widehat{SU(2)}_{N_j-2}$ [14, 15, 16].

We can check the modular invariance of the above partition function.

3.3 GSO dependent part of the partition function : $d = 4$ case

In this subsection, we comment on the $d = 4$ case. To construct the modular invariant partition function in the $d = 4$ case, we combine the four fermions to construct the affine currents $\widehat{SO(2)}_1 \times \widehat{SO(2)}_1$ and label the the Verma module by indices s_{-1} and s_0. Then, the modular invariant partition function can be constructed in almost the same way as the $d = 2, 6$ cases.

First we define the vectors $\underset{\sim}{\mu}$'s and the inner product between them as

$$\begin{aligned}
\underset{\sim}{\mu} &:= (s_{-1}, s_0; s_1, \ldots, s_R; m_1, \ldots, m_R; M),\\
\underset{\sim}{\mu} \bullet \underset{\sim}{\mu}' &:= -\frac{s_{-1}s'_{-1}}{4} - \frac{s_0 s'_0}{4} - \sum_j \frac{s_j s'_j}{4} + \sum_j \frac{m_j m'_j}{2N_j} - \frac{MM'}{2KJ}.
\end{aligned}$$

It is convenient to introduce special vectors $\underset{\sim}{\beta}_0$, $\underset{\sim}{\beta}_j$ and $\underset{\sim}{\beta}_{-1}$

$$\begin{aligned}
\underset{\sim}{\beta}_0 &= (1, 1; 1, \ldots, 1; 1, \ldots, 1; M),\\
\underset{\sim}{\beta}_j &= (0, 2; 0, \ldots, 0, \underset{\overset{\wedge}{s_j}}{2}, 0, \ldots, 0; 0, \ldots, 0; 0), \ (j = 1, \ldots, R),\\
\underset{\sim}{\beta}_{-1} &= (2, 2; 0, \ldots, 0; 0, \ldots, 0; 0).
\end{aligned}$$

Using these vectors, we can construct the building blocks $f^{\lambda}_{\underset{\sim}{\mu}}(\tau)$ as

$$f^{\lambda}_{\underset{\sim}{\mu}}(\tau) := \chi_{s_{-1}}(\tau)\chi_{s_0}(\tau)\chi^{\ell_1,s_1}_m(\tau) \ldots \chi^{\ell_R,s_R}_m(\tau)\Theta_{M,KJ}(\tau)/\eta(\tau),$$

where $\chi_{s_{-1}}(\tau)$ and $\chi_{s_0}(\tau)$ are $\widehat{SO(2)}_1$ characters. Then the GSO conditions and the condition that boundary conditions of the fermionic currents are the same in all sectors are

$$2\underset{\sim}{\beta}_0 \bullet \underset{\sim}{\mu} \in 2\mathbf{Z} + 1, \qquad \underset{\sim}{\beta}_j \bullet \underset{\sim}{\mu} \in \mathbf{Z}, \qquad \underset{\sim}{\beta}_{-1} \bullet \underset{\sim}{\mu} \in \mathbf{Z}, \tag{3.10}$$

and we can construct the modular invariant partition function by the beta method in this case. Next we introduce the function $F^{\lambda}_{\underset{\sim}{\mu}}(\tau)$ as

$$F^{\lambda}_{\underset{\sim}{\mu}}(\tau) = \sum_{b_0 \in \mathbf{Z}_{2K}, b_j \in \mathbf{Z}_2, b_{-1} \in \mathbf{Z}_2} (-1)^{b_0 + s_0} f^{\lambda}_{\underset{\sim}{\mu} + b_0 \underset{\sim}{\beta_0} + \sum_j b_j \underset{\sim}{\beta_j} + b_{-1} \underset{\sim}{\beta_{-1}}}(\tau),$$

then we obtain the GSO dependent part of the modular invariant partition function Z_{GSO}

$$Z_{GSO}(\tau, \bar{\tau}) = \frac{1}{4^R \times 4K} \sum_{\lambda, \bar{\lambda}, \underset{\sim}{\mu}}^{\text{even,beta}} L_{\lambda \bar{\lambda}} F^{\lambda}_{\underset{\sim}{\mu}}(\tau) \bar{F}^{\bar{\lambda}}_{\underset{\sim}{\mu}}(\bar{\tau}).$$

We can check the modular invariance of the above partition function.

4 Elliptic genus

In this section, we calculate the elliptic genus of the theory [17]. The definition of the elliptic genus is

$$Z(\tau, \bar{\tau}, z) := \mathrm{Tr}_{RR}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} y^{J_0},$$

where the trace is taken for the RR states, and $y = \exp(2\pi i z)$. This elliptic genus has the following modular properties;

$$\begin{aligned} Z(\tau+1, \bar{\tau}+1, z) &= Z(\tau, \bar{\tau}, -z) = Z(\tau, \bar{\tau}, z), \\ Z(-1/\tau, -1/\bar{\tau}, z/\tau) &= \mathbf{e}\left[\frac{\hat{c}}{2}\frac{z^2}{\tau}\right] Z(\tau, \bar{\tau}, z). \end{aligned} \tag{4.1}$$

Here, we omit the contribution from the flat space time and consider only the internal part describing the Calabi-Yau n-fold X. We calculate its elliptic genus and the Witten index, and discuss its geometrical interpretation.

Let us consider again "the criticality condition", in other words "the Calabi-Yau condition". Here the total $\hat{c}$ should be n because we want the theory that describes a Calabi-Yau n-fold. Therefore, the total $\hat{c}$ of the $\mathcal{N} = 2$ Liouville and the minimal models should satisfy the relations

$$n = \hat{c} = 1 + Q^2 + \sum_j \frac{N_j - 2}{N_j}. \tag{4.2}$$

We introduce the following vectors with R components $\{m_j\}$, and the inner product between them as

$$\begin{aligned} \nu &:= (m_1, \ldots, m_R), \\ \nu \bullet \nu' &:= \sum_j \frac{m_j m'_j}{2N_j}. \end{aligned}$$

We also introduce the special vector γ_0 with all components 2

$$\gamma_0 := (2, \ldots, 2).$$

With these notations, the condition (4.2) becomes

$$Q^2 = n - 1 - R + \gamma_0 \bullet \gamma_0.$$

Next we let $N := \mathrm{lcm}(N_j)$, and define J as

$$\frac{2J}{N} := Q^2. \tag{4.3}$$

In this paper, we concentrate only the case $(n-1-R)$ is even, then in this case, J is an integer.

Because we want a Calabi-Yau CFT, we have to pick up only the states with integral $U(1)$ charges. This condition is realized as the condition

$$\gamma_0 \bullet \nu + pQ \in \mathbf{Z}. \tag{4.4}$$

From this, p can be written with an arbitrary integer u

$$p = \frac{1}{Q}\left(u - \gamma_0 \bullet \nu\right).$$

Following the same manner as in the previous section, we let $u = 2Jv + w$ and sum up for all integer v. It leads to the theta function

$$\sum_{v \in \mathbf{Z}} q^{\frac{1}{2}p^2} y^{pQ} = \Theta_{N(w-\gamma_0\bullet\nu),NJ}(\tau, 2z/N).$$

Note that $N(w - \gamma_0 \bullet \nu)$ is an integer and $\Theta_{N(w-\gamma_0\bullet\nu),NJ}(\tau, 2z/N)$ has good modular properties.

Collecting these, we define the building blocks $g^{\lambda}_{\underset{\sim}{\nu}}$ as

$$g^{\lambda}_{\underset{\sim}{\nu}}(\tau, z) := \sum_{s_0, s_j} \chi^{\lambda}_{\mu}(\tau, z) \frac{\Theta_{M,NJ}(\tau, 2z/N)}{\eta(\tau)} (-1)^{-\frac{s_0}{2} - \sum_j \frac{s_j}{2} + \gamma_0\bullet\nu + \frac{M}{N}},$$

where $\underset{\sim}{\nu} := (\nu, M)$. In the sign $(-1)^{J_0} = (-1)^{-\frac{s_0}{2} - \sum_j \frac{s_j}{2} + \gamma_0\bullet\nu + \frac{M}{N}}$, the part $(-\frac{s_0}{2} - \sum_j \frac{s_j}{2})$ represents contributions of ordinary $U(1)$ charges from the indices s_0, s_j, and the rest $\gamma_0 \bullet \nu + \frac{M}{N} = w = u - 2Jv$ reflects contributions from the indices m_j and S^1 momentum.

Let us define the inner product between $\underset{\sim}{\nu}$ and $\underset{\sim}{\nu}'$ as

$$\underset{\sim}{\nu} \bullet \underset{\sim}{\nu}' := \nu \bullet \nu' - \frac{MM'}{2NJ},$$

and the special vector

$$\underset{\sim}{\gamma_0} = (\gamma_0, -2J).$$

We also introduce the functions I^ℓ_m and I^λ_ν

$$I^\ell_m(\tau,z) := \chi^{\ell,1}_m(\tau,z) - \chi^{\ell,-1}_m(\tau,z),$$
$$I^\lambda_\nu(\tau,z) := I^{\ell_1}_{m_1}(\tau,z)\dots I^{\ell_R}_{m_R}(\tau,z).$$

With these notations, the building blocks $g^\lambda_{\underset{\sim}{\nu}}$ can be written as

$$g^\lambda_{\underset{\sim}{\nu}}(\tau,z) = \frac{\theta_1(\tau,z)}{\eta(\tau)} I^\lambda_\nu(\tau,z) \frac{\Theta_{M,NJ}(\tau,2z/N)}{\eta(\tau)} (-1)^{\underset{\sim}{\gamma_0}\bullet\underset{\sim}{\nu}},$$

where we omit the overall irrelevant phase. The condition (4.4) can be rewritten as

$$\underset{\sim}{\gamma_0} \bullet \underset{\sim}{\nu} \in \mathbf{Z}, \tag{4.5}$$

and again we call this condition "the beta condition".

Now, we construct elliptic genus using the above building blocks $g^\lambda_{\underset{\sim}{\nu}}$ which satisfy the condition (4.5).

Note that if $\underset{\sim}{\nu}$ satisfies the beta condition, then $\underset{\sim}{\nu} + b_0\underset{\sim}{\gamma_0}$ for $b_0 \in \mathbf{Z}$ also satisfies the beta condition, because

$$\underset{\sim}{\gamma_0} \bullet \underset{\sim}{\gamma_0} = \gamma_0 \bullet \gamma_0 - \frac{2J}{N} = -(n-1-R),$$

is an integer. [1] Here we used the definition of J (4.3). So let us define the new functions $G^\ell_{\underset{\sim}{\nu}}$ as follows;

$$G^\ell_{\underset{\sim}{\nu}}(\tau,z) = \sum_{b_0\in\mathbf{Z}_N} g^\lambda_{\underset{\sim}{\nu}+b_0\underset{\sim}{\gamma_0}}(\tau,z),$$

where $\underset{\sim}{\nu}$ satisfies the beta condition (4.4). Then, from the modular properties of $g^\lambda_{\underset{\sim}{\nu}}$

$$g^\lambda_{\underset{\sim}{\nu}}(\tau+1,z) = \mathbf{e}\left[\sum_j \frac{\ell_j(\ell_j+1)}{4N_j} - \frac{1}{2}\underset{\sim}{\nu}\bullet\underset{\sim}{\nu} + \frac{R+1}{8} - \frac{1}{24}\left(\sum_j \frac{N_j-2}{N_j} + 2\right)\right] g^\lambda_{\underset{\sim}{\nu}}(\tau,z),$$
$$g^\lambda_{\underset{\sim}{\nu}}(-1/\tau,z/\tau) = (-i)^R\,\mathbf{e}\left[\frac{n}{2}\frac{z^2}{\tau}\right]$$
$$\times \sum_{\lambda',\underset{\sim}{\nu}'}^{\text{even}} A_{\lambda\lambda'} \frac{1}{\prod_j\sqrt{N_j}} \frac{1}{\sqrt{2NJ}}\,\mathbf{e}\left[\underset{\sim}{\nu}\bullet\underset{\sim}{\nu}'\right] (-1)^{\underset{\sim}{\gamma_0}\bullet(\underset{\sim}{\nu}-\underset{\sim}{\nu}')} g^{\lambda'}_{\underset{\sim}{\nu}'}(\tau,z),$$

[1] Actually, it is an even integer. Remember that we concentrate the case in which $(n-1-R)$ is an even integer.

$G^{\ell}_{\underset{\sim}{\nu}}$ have very good modular properties;

$$G^{\lambda}_{\underset{\sim}{\nu}}(\tau+1,z) = \mathbf{e}\left[\sum_j \frac{\ell_j(\ell_j+1)}{4N_j} - \frac{1}{2}\underset{\sim}{\nu}\bullet\underset{\sim}{\nu} + \frac{R+1}{8} - \frac{1}{24}\left(\sum_j \frac{N_j-2}{N_j}+2\right)\right] G^{\lambda}_{\underset{\sim}{\nu}}(\tau,z),$$

$$G^{\lambda}_{\underset{\sim}{\nu}}(-1/\tau,z/\tau) = (-i)^R\,\mathbf{e}\left[\frac{n}{2}\frac{z^2}{\tau}\right]$$
$$\times \sum_{\lambda',\underset{\sim}{\nu}'}^{\text{even,beta}} A_{\lambda\lambda'}\frac{1}{\prod_j\sqrt{N_j}}\frac{1}{\sqrt{2NJ}}\,\mathbf{e}\left[\underset{\sim}{\nu}\bullet\underset{\sim}{\nu}'\right](-1)^{\underset{\sim}{\gamma_0}\bullet(\underset{\sim}{\nu}-\underset{\sim}{\nu}')} G^{\lambda'}_{\underset{\sim}{\nu}'}(\tau,z).$$

Using these functions, we obtain the elliptic genus in the following form;

$$Z(\tau,\bar{\tau},z) = \frac{1}{2^R N}\frac{1}{\sqrt{\tau_2}|\eta(\tau)|^2}\sum_{\lambda,\bar{\lambda},\underset{\sim}{\nu}}^{\text{even,beta}} L_{\lambda\bar{\lambda}}G^{\lambda}_{\underset{\sim}{\nu}}(\tau,z)\bar{G}^{\bar{\lambda}}_{\underset{\sim}{\nu}}(\bar{\tau},0).$$

Here $L_{\lambda\bar{\lambda}}$ is the product of $\widehat{SU(2)}$ modular invariants, and the factor $1/\sqrt{\tau_2}|\eta(\tau)|^2$ is contribution of ϕ. We can check that the above elliptic genus has the right modular properties (4.1) with $\hat{c}=n$.

Actually, this elliptic genus is 0 because it has an overall factor $\bar{\theta}_1(\bar{\tau},0)=0$.

4.1 Hodge Number and Witten index

To get some nontrivial information from the above elliptic genus, we factor out the trivial parts and define $\hat{Z}$ by the equations

$$Z(\tau,\bar{\tau},z) = \frac{\theta_1(\tau,z)\bar{\theta}_1(\tau,0)}{\sqrt{\tau_2}|\eta(\tau)|^6}\hat{Z}(\tau,\bar{\tau},z),$$
$$\hat{Z}(\tau,\bar{\tau},z) = \frac{1}{2^{-R}N}\sum_{\lambda,\bar{\lambda},\underset{\sim}{\nu}}^{\text{even,beta}} L_{\lambda\bar{\lambda}}\hat{G}^{\lambda}_{\underset{\sim}{\nu}}(\tau,z)\bar{\hat{G}}^{\bar{\lambda}}_{\underset{\sim}{\nu}}(\bar{\tau},0),$$
$$\hat{G}^{\lambda}_{\underset{\sim}{\nu}}(\tau,z) = \sum_{b_0\in\mathbf{Z}_N}\hat{g}^{\lambda}_{\underset{\sim}{\nu}+b_0\underset{\sim}{\gamma_0}}(\tau,z),$$
$$\hat{g}^{\lambda}_{\underset{\sim}{\nu}}(\tau,z) = I^{\lambda}_{\nu}(\tau,z)\Theta_{M,NJ}(\tau,2z/N)(-1)^{\underset{\sim}{\gamma_0}\bullet\underset{\sim}{\nu}},$$

Now, we take the limit $\tau\to i\infty$ and consider the ground states. In this limit, $\Theta_{M,NJ}$ becomes

$$\Theta_{M,NJ}(i\infty,z) = \delta_M^{\text{mod NJ}},$$

so, the $\hat{G}$'s can be evaluated as

$$\hat{G}^{\lambda}_{\underset{\sim}{\nu}} = \begin{cases} I^{\lambda}_{\nu+\frac{M}{2J}\gamma_0}(i\infty,z) & (M\equiv 0 \mod 2J), \\ 0 & (\text{others}). \end{cases}$$

Then, $\hat{Z}$ is expressed in the formula

$$\lim_{\tau\to i\infty} \hat{Z} = \frac{1}{2^R N} \sum_{\lambda,\bar{\lambda},\underset{\sim}{\nu}}^{\text{even,beta}} \delta_M^{\text{mod 2J}} L_{\lambda\bar{\lambda}} I^{\lambda}_{\nu+\frac{M}{2J}\gamma_0}(i\infty, z) \bar{I}^{\bar{\lambda}}_{\nu+\frac{M}{2J}\gamma_0}(-i\infty, z).$$

When we replace the $\nu+\frac{M}{2J}\gamma_0$ by ν, then we can perform the sum of $M \in \mathbf{Z}_{2NJ}$. Moreover, from the fact

$$I^{\ell_j}_{m_j}(-i\infty, z) = \delta^{\text{mod 2N}_\text{j}}_{m_j-\ell_j-1}\, y^{\frac{\ell+1}{N}-\frac{1}{2}} - \delta^{\text{mod 2N}_\text{j}}_{m_j+\ell_j+1}\, y^{-\frac{\ell+1}{N}+\frac{1}{2}},$$

it can be seen that the even condition $\ell_j + m_j \equiv 1 \mod 2$ is included in this factor. We obtain a formula of the $\hat{Z}$ in this limit

$$\lim_{\tau\to i\infty} \hat{Z} = \frac{1}{2^R} \sum_{\nu}^{\text{beta}} \prod_j \left[\sum_{\ell_j} L^{(N_j)}_{\ell_j\bar{\ell}_j} I^{\ell_j}_{m_j}(i\infty, z) I^{\bar{\ell}_j}_{m_j}(-i\infty, z) \right].$$

So far, we treat rather general cases, but from now, we take an example and restrict ourselves to calculations in the example. We consider the example which satisfies all the following conditions.

- All minimal models are A type. So, $L_{\lambda\bar{\lambda}} = \delta_{\lambda\bar{\lambda}}$.
- $R = n + 1$.
- $N_1 = N_2 = \cdots = N_R = N$.

In other words, this example is the case where the associated Calabi-Yau manifold is the hypersurface of the form

$$z_1^N + z_2^N + \cdots + z_{n+1}^N = 0 \text{ in } \mathbf{C}^{n+1}. \quad (4.6)$$

We can write the finite distance condition as $N < n + 1$, which is equivalent to the condition that first Chern number of $X/\mathbf{C}^\times$ is positive. In this case, nontrivial factor of the elliptic genus can be calculated as

$$\begin{aligned} \lim_{\tau\to i\infty} \hat{Z} &= \frac{1}{2^R} \sum_{\nu}^{\text{beta}} \prod_j \left[\sum_{\ell_j} \left(\delta^{\text{mod 2N}_\text{j}}_{m_j-\ell_j-1}\, y^{\frac{\ell+1}{N}-\frac{1}{2}} - \delta^{\text{mod 2N}_\text{j}}_{m_j+\ell_j+1}\, y^{-\frac{\ell+1}{N}+\frac{1}{2}} \right) \left(\delta^{\text{mod 2N}_\text{j}}_{m_j+\ell_j+1} - \delta^{\text{mod 2N}_\text{j}}_{m_j-\ell_j-1} \right) \right] \\ &= \frac{y^{-\frac{n+1}{2}}}{2^R} \sum_{\nu}^{\text{beta}} \prod_j \left[\sum_{\ell_j} \left(\delta^{\text{mod 2N}_\text{j}}_{m_j-\ell_j-1}\, y^{\frac{\ell+1}{N}} + \delta^{\text{mod 2N}_\text{j}}_{m_j+\ell_j+1}\, y^{-\frac{\ell+1}{N}+1} \right) \right], \end{aligned}$$

When we put $m_j = a_j + Nb_j, \quad (b_j = 0, -1,\ a_j = 0, 1, \ldots N - 1)$, then beta condition becomes

$$\sum_j a_j \equiv 0 \mod N.$$

$n=3$

$N\backslash p$	1	2	3
3	0	6	0
4	1	19	1

$n=4$

$N\backslash p$	1	2	3	4
3	0	5	5	0
4	0	30	30	0
5	1	101	101	1

$n=5$

$N\backslash p$	1	2	3	4	5
3	0	1	20	1	0
4	0	21	141	21	0
5	0	120	580	120	0
6	1	426	1751	426	1

Table 1: The values of the coefficients h_p for $n=3,4,5,\ N=3,\dots,n+1$. We include the $N=n+1$ case in the table, despite it is suppressed by the finite distance condition.

We obtain the $\hat{Z}$ in this limit

$$\lim_{\tau\to i\infty}\hat{Z}=\sum_{p=1}^{n}h_p y^{p-\frac{n+1}{2}},$$

where the coefficients h_p of $y^{p-\frac{n+1}{2}}$ are represented as

$$h_p:=\sum_{\substack{a_j=1,\dots,N-1,\\ \sum_j a_j=pN}}1=\sum_{i=0}^{p}(-1)^i\binom{n+1}{i}\binom{(p-i)(N-1)+p-1}{n}.$$

We show several examples of h_p for lower n,N in Table 1. These coefficients h_p seem to coincide with the middle dimensional Hodge numbers of $X/\mathbf{C}^\times$ except for the cohomology elements generated by cup products of a Kähler form of $X/\mathbf{C}^\times$, on which we mention below.

The Witten index can also be calculated as

$$\begin{aligned}\lim_{\tau\to i\infty,z\to 0}\hat{Z}&=\sum_{p=1}^{n}h_p\\ &=(-1)^{n+1}\left[1+\frac{(1-N)^{n+1}-1}{N}\right]\\ &=(-1)^{n+1}\left[n+1+\frac{(1-N)^{n+1}-1}{N}-n\right].\end{aligned}$$

On the other hand, the Euler number of the $(n-1)$-dimensional manifold $X/\mathbf{C}^\times$ is expressed in the formula

$$\chi_Y=n+1+\frac{(1-N)^{n+1}-1}{N}.$$

The Witten index of the CFT almost coincide with the Euler number of $X/\mathbf{C}^\times$. One of the difference of the two is the sign $(-1)^{n+1}$, but this is not relevant. Except this difference of the sign, the Witten index is smaller by n than the Euler number of $X/\mathbf{C}^{\times}$ in our case. This difference may correspond to the cohomology elements generated by cup products of the Kähler form of $X/\mathbf{C}^{\times}$. On X, these cohomology elements are absent because these forms appear when we take the quotient of X by $\mathbf{C}^{\times}$. This seems the reason why the Witten index is smaller by n than the Euler number of $X/\mathbf{C}^{\times}$.

5 Conclusion and Discussion

We construct the toroidal partition function of the string theory described by the combination of an $\mathcal{N}=2$ Liouville theory and multiple $\mathcal{N}=2$ minimal models. This partition function is actually modular invariant, and we can conclude that the theory exists consistently.

This string theory is thought to describe the string on a non-compact singular Calabi-Yau manifold. To check this proposition, we also calculate the elliptic genus of this theory and the Witten index.

The Euler number defined from the Witten index in the CFT seems to be that of the non-vanishing elements of the cohomology. In the case of a singular manifold, there are vanishing elements of the cohomology, which is supported on the singular points and reflect the structure of singularities.

The fact that the vanishing elements of the cohomology cannot be seen, is probably related to our method of construction in which we include only the states of "principal continuous series" in the $SL(2)$ theory. If we can include some "discrete series" (but it is difficult[18]) , the structure of the singularities might be seen in the CFT.

We may not be able to use our result to analyze the structure of the singularities, but we can use it to analyze the string on the positively curved manifold $X/\mathbf{C}^{\times}$. Especially it is interesting to analyze the D-branes wrapped on infinite cycle in this non-compact Calabi-Yau manifold through the recipes of boundary states in the CFT [19, 20, 21] as the case of the ordinary Gepner models [22, 23, 24, 25].

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Appendix A. Theta functions and characters

We use the following notations in this paper.

$$\mathbf{e}[x] := \exp(2\pi i x),$$
$$\delta_m^{\text{mod N}} := \begin{cases} 1 & (m \equiv 0 \mod N), \\ 0 & (\text{others}), \end{cases}$$

where m and N are integers. The useful formula is

$$\sum_{j \in Z_N} \mathbf{e}\left[\frac{jm}{N}\right] = N\delta_m^{\text{mod N}},$$

where m and N are integers.

The SU(2) classical theta functions is defined as

$$\Theta_{m,k}(\tau, z) = \sum_{n \in \mathbf{Z}} q^{k\left(n+\frac{m}{2k}\right)^2} y^{k\left(n+\frac{m}{2k}\right)},$$

where $q := \mathbf{e}[\tau]\,, y := \mathbf{e}[z]$. The Jacobi's theta functions are also defined as

$$\theta_1(\tau, z) := i \textstyle\sum_{n \in \mathbf{Z}} (-1)^n q^{\left(n-\frac{1}{2}\right)^2} y^{\left(n-\frac{1}{2}\right)}, \quad \theta_2(\tau, z) := \displaystyle\sum_{n \in \mathbf{Z}} q^{\left(n-\frac{1}{2}\right)^2} y^{\left(n-\frac{1}{2}\right)},$$
$$\theta_3(\tau, z) := \textstyle\sum_{n \in \mathbf{Z}} q^{n^2} y^n, \qquad \theta_4(\tau, z) := \displaystyle\sum_{n \in \mathbf{Z}} (-1)^n q^{n^2} y^n.$$

Two kinds of theta functions are related by equations

$$2\Theta_{0,2} = \theta_3 + \theta_4, \ 2\Theta_{1,2} = \theta_2 + i\theta_1,$$
$$2\Theta_{2,2} = \theta_3 - \theta_4, \ 2\Theta_{3,2} = \theta_2 - i\theta_1.$$

The Dedekind η function is represented as an infinite product

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

The character $\chi_s(\tau, z), \ s = 0, 1, 2, 3$ of $\widehat{SO(d)}_1$ for $d/2 \in 2\mathbf{Z} + 1$ can expressed as

$$\chi_0(\tau, z) = \frac{\theta_3(\tau, z)^{d/2} + \theta_3(\tau, z)^{d/2}}{2\eta(\tau)^{d/2}},$$
$$\chi_1(\tau, z) = \frac{\theta_2(\tau, z)^{d/2} + (i\theta_1(\tau, z))^{d/2}}{2\eta(\tau)^{d/2}},$$
$$\chi_2(\tau, z) = \frac{\theta_3(\tau, z)^{d/2} - \theta_3(\tau, z)^{d/2}}{2\eta(\tau)^{d/2}},$$
$$\chi_3(\tau, z) = \frac{\theta_2(\tau, z)^{d/2} - (i\theta_1(\tau, z))^{d/2}}{2\eta(\tau)^{d/2}}.$$

Let us denote the characters of a Verma module (ℓ, m, s) in the level $(N-2)$ minimal model as $\chi_m^{\ell,s}(\tau, z)$. This character satisfies equivalence relations

$$\chi_m^{\ell,s} = \chi_{m+2N}^{\ell,s} = \chi_m^{\ell,s+4} = \chi_{m+N}^{N-2-\ell,s+2}.$$

The explicit form of this character is written in [2].

We collect the modular properties of these functions. Under the T transformations, they behave as

$$\begin{aligned}
&\Theta_{m,k}(\tau+1, z) = \mathbf{e}\left[\frac{m^2}{4k}\right] \Theta_{m,k}(\tau, z), \\
&\theta_1(\tau+1, z) = \mathbf{e}\left[\tfrac{1}{8}\right] \theta_1(\tau, z), \qquad \theta_2(\tau+1, z) = \mathbf{e}\left[\tfrac{1}{8}\right] \theta_2(\tau, z), \\
&\theta_3(\tau+1, z) = \theta_4(\tau, z), \qquad \theta_4(\tau+1, z) = \theta_3(\tau, z), \\
&\eta(\tau+1) = \mathbf{e}[1/24]\, \eta(\tau), \\
&\chi_s(\tau+1, z) = \mathbf{e}\left[\frac{s^2}{8} - \frac{d}{48}\right] \chi_s(\tau, z), \\
&\chi_m^{\ell,s}(\tau+1, z) = \mathbf{e}\left[\frac{\ell(\ell+2)}{4N} - \frac{m^2}{4N} + \frac{s^2}{8} - \frac{N-2}{8N}\right] \chi_m^{\ell,s}(\tau, z),
\end{aligned}$$

and for S transformations, they have modular properties

$$\begin{aligned}
&\Theta_{m,k}(-1/\tau, z/\tau) = \sqrt{-i\tau}\, \mathbf{e}\left[\frac{k}{4}\frac{z^2}{\tau}\right] \sum_{m' \in \mathbf{Z}_{2k}} \frac{1}{\sqrt{2k}}\, \mathbf{e}\left[-\frac{mm'}{2k}\right] \Theta_{m',k}(\tau, z), \\
&\theta_1(-1/\tau, z/\tau) = -i\sqrt{-i\tau}\, \mathbf{e}\left[\frac{1}{2}\frac{z^2}{\tau}\right] \theta_1(\tau, z), \qquad \theta_2(-1/\tau, z/\tau) = \sqrt{-i\tau}\, \mathbf{e}\left[\frac{1}{2}\frac{z^2}{\tau}\right] \theta_4(\tau, z), \\
&\theta_3(-1/\tau, z/\tau) = \sqrt{-i\tau}\, \mathbf{e}\left[\frac{1}{2}\frac{z^2}{\tau}\right] \theta_3(\tau, z), \qquad \theta_4(-1/\tau, z/\tau) = \sqrt{-i\tau}\, \mathbf{e}\left[\frac{1}{2}\frac{z^2}{\tau}\right] \theta_2(\tau, z), \\
&\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau), \\
&\chi_s(-1/\tau, z/\tau) = \mathbf{e}\left[\frac{d}{4}\frac{z^2}{\tau}\right] \sum_{s'=0}^{3} \frac{1}{2}\, \mathbf{e}\left[-\frac{d}{2}\frac{ss'}{4}\right] \chi_{s'}(\tau, z), \\
&\chi_m^{\ell,s}(-1/\tau, z/\tau) = \mathbf{e}\left[\frac{N-2}{2N}\frac{z^2}{\tau}\right] \frac{1}{2\sqrt{N}} \sum_{\ell,m,s}^{\text{even}} A_{\ell\ell'}\, \mathbf{e}\left[-\frac{ss'}{4} + \frac{mm'}{2N}\right] \chi_{m'}^{\ell',s'}(\tau, z), \\
&A_{\ell\ell'} = \sqrt{\frac{2}{N}} \sin\left[\pi\frac{(\ell+1)(\ell'+1)}{N}\right],
\end{aligned}$$

where the sum $\sum_{\ell,m,s}^{\text{even}}$ means that for (ℓ, m, s) with $\ell + m + s \equiv 0 \mod 2$.

We use the notation for a function of τ, z with omit z as

$$f(\tau) := f(\tau, z = 0).$$

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